

A NOTE ON EXTENDED RECURRENT LORENTZIAN MANIFOLDS

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ABSTRACT. Extended recurrent pseudo-Riemannian manifolds were introduced by Mileva Prvanović. We reconsider her work in the light of recent results and show that the manifold is conformally flat, and it is a space of quasi-constant curvature. We also show that an extended recurrent Lorentzian manifold, with time-like associated covector, is a perfect fluid Robertson-Walker space-time. We obtain the equation of state; in $n = 4$ and if the scalar curvature is zero, a model for incoherent radiation is obtained.

Dedicated to the memory of Dr. Mileva Prvanović

1. INTRODUCTION

In 1999 Mileva Prvanović [22] introduced the following differential structure on a pseudo-Riemannian manifold, that she named “extended recurrent manifold”:

$$(1) \quad \begin{aligned} \nabla_i R_{jklm} &= A_i R_{jklm} + (\beta - \psi) A_i G_{jklm} \\ &+ \frac{\beta}{2} [A_j G_{iklm} + A_k G_{jilm} + A_l G_{jkim} + A_m G_{jkli}] \end{aligned}$$

A_i is a closed one-form named “associated covector”, β and ψ are non vanishing scalar functions with $\nabla_j \psi = \beta A_j$, $G_{jklm} = g_{mj}g_{kl} - g_{mk}g_{jl}$.

She proved that the associated covector is a concircular vector: $\nabla_s A_r = f g_{rs} + h A_r A_s$ with scalar functions f and h , and showed that the metric has the warped form

$$(2) \quad ds^2 = (dx^1)^2 + e^\eta g_{\alpha\beta}^* dx^\alpha dx^\beta$$

where $g_{\alpha\beta}^*$ are functions only of x^γ ($\gamma = 2, \dots, n$) and η is a scalar function of x^1 . These properties will be reviewed in Section 2, where we also derive some new ones. In particular we show that an extended recurrent pseudo-Riemannian manifold is conformally flat, and it is a space of quasi constant curvature, according the definition by K. Yano and B.-Y. Chen [5]. In Section 3 we focus on extended recurrent Lorentzian manifolds (space-times). Based on our recent study of Generalized Robertson Walker manifolds, to which the present model eventually belongs, we show that an extended recurrent space-time with time-like associated covector is a perfect fluid Robertson-Walker spacetime. The barotropic equation of state is

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obtained; in the particular case of vanishing scalar curvature, in 4 dimensions, we obtain a model for incoherent radiation.

Throughout the paper we adopt the convention $R_{ij} = R_{imj}{}^m$ and $R = R^m{}_m$ for the Ricci tensor and the scalar curvature, and use the notation $v^2 = v^m v_m$.

2. GENERAL PROPERTIES OF EXTENDED RECURRENT PSEUDO-RIEMANNIAN MANIFOLDS

We review some basic properties of extended recurrent pseudo-Riemannian manifolds exposed in [22]. Furthermore, we prove some new characterizations of such manifolds.

Following the procedure in [22], by contracting (1) with g^{jm} it is

$$(3) \quad \nabla_i R_{kl} = A_i [R_{kl} - g_{kl}(n\beta - (n-1)\psi)] - \frac{\beta}{2}(n-2)(A_k g_{il} + A_l g_{ik}).$$

Contracting again (3) with g^{kl} we obtain

$$(4) \quad \nabla_i R = A_i [R - (n^2 + n - 2)\beta + n(n-1)\psi].$$

On the other hand, by the second Bianchi identity for the Riemann tensor it is $A_i(R_{jklm} - \psi G_{jklm}) + A_j(R_{kilm} - \psi G_{kilm}) + A_k(R_{ijlm} - \psi G_{ijlm}) = 0$. Contracting this with g^{im} it is

$$(5) \quad R_{jklm} A^m = A_k [R_{jl} + \psi(n-2)g_{jl}] - A_j [R_{kl} + \psi(n-2)g_{kl}].$$

and contracting (5) with g^{kl} we obtain

$$(6) \quad R_{jm} A^m = \frac{1}{2} A_j [R + \psi(n-2)(n-1)].$$

The components of the Weyl conformal curvature tensor are [19]:

$$(7) \quad C_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-2}(g_{jm}R_{kl} - g_{km}R_{jl} + R_{jm}g_{kl} - R_{km}g_{jl}) - \frac{g_{jm}g_{kl} - g_{km}g_{jl}}{(n-1)(n-2)}R$$

By taking the covariant derivative of (7) and inserting (4) and (3) we infer that

$$(8) \quad \nabla_i C_{jklm} = A_i C_{jklm}$$

Now, (5), (6) are used to evaluate $A_m C_{jkl}{}^m$:

$$(9) \quad A_m C_{jkl}{}^m = \frac{n-3}{n-2} \left[A_k \left(R_{jl} - \frac{R - \psi(n-1)(n-2)}{2(n-1)} g_{jl} \right) - A_j \left(R_{kl} - \frac{R - \psi(n-1)(n-2)}{2(n-1)} g_{kl} \right) \right]$$

Next, consider Lovelock's identity ([14] page 289):

$$\nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m = -R_{im} R_{jkl}{}^m - R_{jm} R_{kil}{}^m - R_{km} R_{ijl}{}^m$$

The evaluation of $\nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m$ with the aid of (3) gives zero, therefore it is $R_{im} R_{jkl}{}^m + R_{jm} R_{kil}{}^m + R_{km} R_{ijl}{}^m = 0$. By taking the covariant derivative ∇_s of the last expression and contracting with g^{is} , after long calculations, it is inferred that (provided $\beta \neq 0$ and $n > 3$)

$$(10) \quad A_j \left[R_{kl} - g_{kl} \frac{R - \psi(n-1)(n-2)}{2(n-1)} \right] = A_k \left[R_{jl} - g_{jl} \frac{R - \psi(n-1)(n-2)}{2(n-1)} \right]$$

From (10) and (9) immediately it is $\nabla_m C_{jkl}^m = A_m C_{jkl}^m = 0$.
The second Bianchi identity for the Weyl tensor is (see [1])

$$\begin{aligned} \nabla_i C_{jkl}^m + \nabla_j C_{kil}^m + \nabla_k C_{ijl}^m = \frac{1}{n-3} \Big[& \delta_j^m \nabla_p C_{kil}^p + \delta_k^m \nabla_p C_{ijl}^p \\ & + \delta_i^m \nabla_p C_{jkl}^p + g_{kl} \nabla_p C_{ji}^{mp} + g_{il} \nabla_p C_{kj}^{mp} + g_{jl} \nabla_p C_{ik}^{mp} \Big] \end{aligned}$$

For a conformally recurrent manifold it becomes

$$(11) \quad \begin{aligned} A_i C_{jklm} + A_j C_{kilm} + A_k C_{ijlm} = \frac{A^p}{n-3} \Big[& g_{mj} C_{kilp} + g_{mk} C_{ijlp} \\ & + g_{mj} C_{jklp} + g_{kl} C_{jimp} + g_{il} C_{kjmp} + g_{jl} C_{ikmp} \Big] = 0 \end{aligned}$$

because $A_p C_{jkl}^p = 0$. Thus in our case it is $A_i C_{jklm} + A_j C_{kilm} + A_k C_{ijlm} = 0$ from which $A^2 C_{jklm} = 0$. Therefore, if $A^2 \neq 0$, the manifold is conformally flat: $C_{jklm} = 0$. Moreover if $A^2 \neq 0$ eq.(10) readily rewrites as:

$$(12) \quad 2(n-1)R_{kl} - g_{kl}(R - \psi(n-1)(n-2)) = \frac{A_k A_l}{A^2} (n-2)[R + \psi n(n-1)]$$

and shows that the space is quasi-Einstein (see for example [8, 10, 11, 12, 20]):

$$(13) \quad R_{kl} = a g_{kl} + b \frac{A_k A_l}{A^2}, \quad a = \frac{R - \psi(n-1)(n-2)}{2(n-1)}, \quad b = \frac{n-2}{2(n-1)} [R + \psi n(n-1)]$$

Inserting this in (7) with $C_{jklm} = 0$ gives the Riemann tensor:

$$(14) \quad \begin{aligned} R_{jklm} = \frac{b}{n-2} \Big[& -g_{jm} \frac{A_k A_l}{A^2} + g_{km} \frac{A_j A_l}{A^2} - g_{kl} \frac{A_j A_m}{A^2} + g_{jl} \frac{A_k A_m}{A^2} \Big] \\ & + \psi (g_{jm} g_{kl} - g_{jl} g_{km}). \end{aligned}$$

Eq.(14) characterizes the “manifolds of quasi constant curvature”, introduced by Chen and Yano in 1972 [5]. We thus proved the following

Theorem 2.1. *An $n \geq 3$ dimensional extended recurrent pseudo-Riemannian manifold is conformally flat and is a space of quasi-constant curvature.*

Note that the hypothesis $\nabla_j \psi = A_j \beta$ is not used in the proof of Theorem 2.1. As shown in [22], the covariant derivative ∇_s of (12) and the condition $\nabla_j \psi = A_j \beta$ imply that

$$\nabla_s A_r = f g_{rs} + \omega_s A_r$$

where $f = -\frac{(n-1)\beta}{R+n(n-1)\psi} A^2$, $\omega_s = h A_s$, $h = \frac{A^j A^l \nabla_j A_l}{A^4} + \frac{(n-1)\beta}{R+n(n-1)\psi}$. By showing $\nabla_s h = \mu A_s$ it follows that ω_s is closed (i.e. A_j is a proper concircular vector). Based on the works [29, 30] by Yano, Prvanovic in [22] concluded that the metric has the warped form (2).

3. EXTENDED RECURRENT SPACE-TIMES

In this section we consider extended recurrent Lorentzian manifolds (i.e. space-times) with a time-like associated covector ($A^2 < 0$). We prove it that it is a Robertson-Walker space-time. For this, we need a generalization of such spaces:

An $n \geq 3$ dimensional Lorentzian manifold is named generalized Robertson-Walker space-time (for short GRW) if the metric may take the shape:

$$(15) \quad ds^2 = -(dx^1)^2 + q(x^1)^2 g_{\alpha\beta}^*(x^2, \dots, x^n) dx^\alpha dx^\beta,$$

A GRW space-time is thus the warped product $1 \times q^2 M^*$ ([2, 3, 25, 26]) where M^* is a $(n-1)$ -dimensional Riemannian manifold. If M^* is a 3-dimensional Riemannian manifold of constant curvature, the space-time is called Robertson-Walker space-time. GRW space-times are thus a wide generalization of Robertson-Walker space-times on which standard cosmology is modelled and include the Einstein-de Sitter space-time, the Friedman cosmological models, the static Einstein space-time, and the de Sitter space-time. They are inhomogeneous space-times admitting an isotropic radiation (see Sánchez [25]). We refer to the works by Romero et al. [23, 24], Sánchez [25] and Gutiérrez and Olea [13] for an exhaustive presentation of geometric and physical properties.

Recently, perfect fluids with the condition $\nabla_m C_{jkl}^m = 0$ were studied in [15] and [16], where the authors showed that such spaces are GRW space-times.

The following deep result was proved by Bang Yen Chen, in ref.[4] (for similar results see also the works by Yano [29, 30], Prvanović [21], and the recent paper [9]).

Theorem 3.1 (Chen). *Let (M, g) an $n \geq 3$ dimensional Lorentzian manifold. The space-time is a generalized Robertson-Walker space-time if and only if it admits a time-like vector of the form $\nabla_k X_j = \rho g_{kj}$.*

In the previous section we reviewed Prvanovic's result that the associate covector is concircular, $\nabla_j A_k = f g_{jk} + \omega_j A_k$, with $\omega_j = h A_j$ being a closed one-form. In this case $\omega_j = \nabla_j \sigma$ for a suitable scalar function.

If the associated covector is time-like, i.e. $A^2 < 0$ (with Lorentzian signature), then it can be rescaled to a time-like vector $X_k = A_k e^{-\sigma}$ such that $\nabla_j X_k = \rho g_{kj}$: in fact it is $\nabla_j X_k = (\nabla_j A_k - \omega_j A_k) e^{-\sigma} = (f e^{-\sigma}) X_k$. By Chen's theorem 3.1, the space is a GRW space-time (see [15, 16]).

Thus for $A^2 < 0$ Prvanović's model (1) is a quasi-Einstein GRW space-time with $C_{jklm} = 0$. It is well known (see [7]) that in this case the fiber is a space of constant curvature and the GRW space-time reduces to an ordinary Robertson-Walker model. Moreover in the region $A^2 < 0$, on defining $u_k = A_k / \sqrt{-A^2}$, it is $u^2 = -1$ and the Ricci tensor (13) becomes $R_{kl} = a g_{kl} - b u_k u_l$. With this form of the Ricci tensor, a Lorentzian manifold is named perfect fluid space-time [15].

Theorem 3.2. *An $n > 3$ dimensional extended recurrent Lorentzian manifold with $A^2 < 0$ is a Robertson-Walker space-time.*

Remark 3.3. *In [17], we proved that for a GRW space-time the condition $\nabla_m C_{jkl}^m = 0$ is equivalent to have $R_{kl} = a g_{kl} + b \frac{X_k X_l}{X^2}$ where X_j is the concircular vector of Chen's theorem. Prvanović's model matches these conditions.*

Some physical consequences are now outlined. Let (M, g) be an n -dimensional Lorentzian manifold equipped with Einstein's field equations without cosmological constant,

$$(16) \quad R_{kl} - \frac{1}{2} R g_{kl} = \kappa T_{kl}$$

$\kappa = 8\pi G$ is Einstein's gravitational constant (in units $c = 1$) and T_{kl} is the stress-energy tensor describing the matter content of the space-time (see for example

[6, 18, 28]). Eq.(16) is used to evaluate T_{kl} obtaining:

$$\kappa T_{kl} = -\frac{n-2}{2(n-1)}[R + \psi(n-1)](g_{kl} - u_k u_l)$$

We recognize a perfect fluid stress-energy tensor $T_{kl} = (p + \mu)u_k u_l + pg_{kl}$, being p the isotropic pressure, μ the energy density and u_j the fluid flow velocity. It is

$$(17) \quad \kappa p = -\frac{n-2}{2(n-1)}[R + \psi(n-1)], \quad \kappa \mu = -\frac{1}{2}\psi(n-1)(n-2)$$

One reads that the (non constant) function ψ controls the energy density of the perfect fluid (then it must be negative). An equation of state can be written:

$$p = \frac{\mu}{n-1} - \frac{n-2}{2(n-1)} \frac{R}{\kappa}.$$

In $n = 4$ dimensions with the particular choice $R = 0$, we have a model for incoherent radiation: $p = \mu/3$ [27] (a superposition of waves of a massless field with random propagation directions).

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